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Estimates of Solutions to the Linear Differential Equations of Neutral Type with Several Delays of the Argument

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Abstract—Under study is a system of linear differential equations of neutral type with several delays of the argument. We obtain the conditions on the matrix coefficients of the system under which all solutions decrease with an exponential rate at infinity. Using some functionals of Lyapunov–Krasovskii type, the uniform estimates of solutions are established.

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In the paper we consider a system of linear differential equations with several delays of the argument

$$\frac{d}{dt}\left(y(t) + Dy(t-\tau)\right) = Ay(t) + By(t-\tau) + \sum_{j=1}^{p} B_j y(t-\tau_j), \qquad t > \tau > \tau_j > 0, \tag{1}$$

where A, B, B_j, and D are constant $(n \times n)$-matrices; τ and $\tau_j > 0$ are delay parameters. For $D \neq 0$ such equations are usually called *equations of neutral type*. We study the conditions on the matrix coefficients of (1) under which all solutions decrease with an exponential rate as $t \to \infty$. The power of the exponents characterizing the decay rate is calculated using estimates of some functionals of Lyapunov–Krasovskii type.

1. THE PROBLEM OF SOLUTION STABILITY TO THE EQUATIONS OF NEUTRAL TYPE

For some classes of systems of linear delay differential equations, some conditions are well-known for the asymptotic stability of the zero solution that are formulated in terms of the roots of quasipolynomials or in the form of matrix inequalities (for example, see [1–4]). In particular, it follows from those results that under certain conditions all solutions of linear systems with constant coefficients stabilize with an exponential rate as $t \to \infty$. In practice, however, calculation the power of the exponents characterizing the decay rate of solutions at infinity is a challenge. In a number of works (for instance, see [5–10]), some ways were proposed for obtaining the explicit estimates of stabilization rate of the solutions. These methods are based on the use of some modifications of the Lyapunov–Krasovskii functionals. In particular, in [8–10], to obtain some uniform estimates of solutions to the systems of the form (1) with one delay parameter $\tau > 0$ the following functional was used:

$$v(t,y) = \langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds, \tag{2}$$

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where

$$H = H^* > 0, \qquad K(s) = K^*(s) \in C^1[0,\tau], \qquad K(s) > 0, \qquad \frac{d}{ds}K(s) < 0, \qquad s \in [0,\tau]. \tag{3}$$

We give a result for a solution to the initial value problem

$$\begin{gathered} \frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau), \qquad t > \tau, \\ y(t) = \varphi(t) \ \text{ for } \ t \in [0,\tau], \qquad y(\tau+0) = \varphi(\tau), \qquad \varphi(t) \in C^1[0,\tau], \end{gathered} \tag{4}$$

obtained by the use of the functional (2) and follows from [10].

Theorem 1. *Suppose that there exist matrices* $H, K(s) \in C^1[0,\tau]$ *satisfying* (3) *and the matrix*

$$C = -\begin{pmatrix} HA + A^*H + K(0) & HB + A^*HD \\ B^*H + D^*HA & D^*HB + B^*HD - K(\tau) \end{pmatrix}$$

is positive definite. Let $c_1 > 0$ *be the minimal eigenvalue of the matrix* C *and* $k > 0$ *be the maximal number such that*

$$\frac{d}{ds}K(s) + kK(s) \le 0, \qquad s \in [0,\tau].$$

Then for a solution $y(t)$ *to the initial value problem* (4) *the following estimates hold:*

(1) *If* $\|D\| < q^{-1}$ *with* $q = \exp(\gamma\tau/(2\|H\|))$ *then*

$$\|y(t)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)(M(1-q\|D\|)^{-1} + 1)\Phi, \qquad t > \tau,$$

where

$$\gamma = \min\left\{\frac{c_1}{1+\|D\|^2}, k\|H\|\right\},$$

$$M = \sqrt{\|H^{-1}\|(2\,\|H\|\,(1+\|D\|^2) + \tau K)}, \qquad K = \max_{s\in[0,\tau]}\|K(s)\|, \qquad \Phi = \max_{s\in[0,\tau]}\|\varphi(s)\|.$$

(2) *If* $\|D\| = q^{-1}$ *then*

$$\|y(t)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(M\frac{t}{\tau} + 1\right)\Phi, \qquad t > \tau.$$

(3) *If* $q^{-1} < \|D\| \le 1$ *then*

$$\|y(t)\| \le \exp\left(\frac{(t-\tau)}{\tau}\ln\|D\|\right)\left(Mq(q\|D\| - 1)^{-1} + 1\right)\Phi, \qquad t > \tau.$$

Analyzing the proof of the estimates in Theorem 1 in [10], it is easy to generalize this result changing the condition $\|D\| < 1$ for the condition of belonging the spectrum of the matrix D to a unit circle.

In the present paper, we continue the studies of [8–10] and, under certain conditions on the matrix coefficients of (1), obtain the uniform estimates of the norms of solutions to the initial value problem

$$\begin{gathered} \frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau) + \sum_{j=1}^{p} B_j y(t-\tau_j), \qquad t > \tau > \tau_j > 0, \\ y(t) = \varphi(t) \ \text{ for } \ t \in [0,\tau], \qquad y(\tau+0) = \varphi(\tau), \qquad \varphi(t) \in C^1[0,\tau]. \end{gathered} \tag{5}$$

It will follow from these estimates that solutions to (5) decrease with an exponential rate as $t \to \infty$.

In the sequel, we will assume that spectrum $\sigma(D)$ of the matrix D belongs to the unit circle.

While estimating the solutions to (5), we will use the functional of Lyapunov–Krasovskii type:

$$V(t,y) = \langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle$$
$$+\int\limits_{t-\tau}^{t}\langle K(t-s)y(s),y(s)\rangle\,ds + \sum_{j=1}^{p}\int\limits_{t-\tau_j}^{t}\langle K_j(t-s)y(s),y(s)\rangle\,ds, \qquad t\ge\tau, \quad (6)$$

where the matrices $H, K(s)\in C^1[0,\tau]$ satisfy (3) and the matrices $K_j(s)=K_j^*(s)\in C^1[0,\tau_j]$ are such that

$$K_j(s)>0, \qquad \frac{d}{ds}K_j(s)<0, \qquad s\in[0,\tau_j], \qquad j=1,\dots,p. \quad (7)$$

2. ESTIMATES OF THE MODIFIED LYAPUNOV–KRASOVSKII FUNCTIONAL

Here we obtain an inequality for the functional (6) considered on solutions to the initial value problem (5). In Section 3, this inequality will be used to obtain some uniform estimates of solutions to the system (1).

Theorem 2. *Let there exist matrices $H, K(s)\in C^1[0,\tau]$ and the matrices $K_j(s)\in C^1[0,\tau_j]$, $j=1,\dots,p$, satisfying the conditions* (3) *and* (7) *respectively. Moreover, assume that, for some $k>0$ and $k_j>0$, the inequalities hold*

$$\frac{d}{ds}K(s)+kK(s)\le 0, \qquad s\in[0,\tau],$$
$$\frac{d}{ds}K_j(s)+k_jK_j(s)\le 0, \qquad s\in[0,\tau_j], \quad j=1,\dots,p.$$

Suppose that the compound matrix

$$C=-\begin{pmatrix} HA+A^*H+K(0)+\sum\limits_{j=1}^{p}K_j(0) & HB+A^*HD & HB_1 & \dots & HB_p \\ B^*H+D^*HA & D^*HB+B^*HD-K(\tau) & D^*HB_1 & \dots & D^*HB_p \\ B_1^*H & B_1^*HD & -K_1(\tau_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_p^*H & B_p^*HD & 0 & \dots & -K_p(\tau_p) \end{pmatrix}$$

is nonnegative definite and for every $(p+2)n$-dimensional vector

$$U=\left(u^\top,(u^0)^\top,(u^1)^\top,\dots,(u^p)^\top\right)^\top$$

the following holds:

$$\langle CU,U\rangle \ge c(\|u\|^2+\|u^0\|^2), \quad (8)$$

where $c>0$ is a constant. Then, for the functional (6) *considered on a solution to the initial value problem* (5)*, we have the estimate*

$$V(t,y)\le \exp\left(-\frac{\gamma(t-\tau)}{\|H\|}\right)\Bigg(\langle H(\varphi(\tau)+D\varphi(0)),(\varphi(\tau)+D\varphi(0))\rangle$$
$$+\int\limits_{0}^{\tau}\langle K(\tau-s)\varphi(s),\varphi(s)\rangle\,ds+\sum_{j=1}^{p}\int\limits_{\tau-\tau_j}^{\tau}\langle K_j(\tau-s)\varphi(s),\varphi(s)\rangle\,ds\Bigg), \qquad t>\tau, \quad (9)$$

where

$$\gamma=\min\left\{\frac{c}{1+\|D\|^2},\ k\|H\|,\ k_1\|H\|,\dots,k_p\|H\|\right\}. \quad (10)$$

Proof. Let $y(t)$ be a solution to (5). Consider the functional (6) on this solution. Differentiating it, after simple manipulations we have

$$\begin{aligned}\frac{d}{dt}V(t,y) \equiv &\left\langle \left(HA + A^*H + K(0) + \sum_{j=1}^{p} K_j(0)\right) y(t), y(t) \right\rangle \\ &+ \langle (HB + A^*HD) y(t-\tau), y(t)\rangle + \langle (B^*H + D^*HA) y(t), y(t-\tau)\rangle \\ &+ \langle (D^*HB + B^*HD - K(\tau)) y(t-\tau), y(t-\tau)\rangle + \sum_{j=1}^{p} \langle HB_j y(t-\tau_j), y(t)\rangle \\ &+ \sum_{j=1}^{p} \langle B_j^* H y(t), y(t-\tau_j)\rangle + \sum_{j=1}^{p} \langle D^*HB_j y(t-\tau_j), y(t-\tau)\rangle \\ &+ \sum_{j=1}^{p} \left\langle B_j^* HD y(t-\tau), y(t-\tau_j)\right\rangle - \sum_{j=1}^{p} \langle K_j(\tau_j) y(t-\tau_j), y(t-\tau_j)\rangle \\ &+ \int_{t-\tau}^{t} \left\langle \frac{d}{dt} K(t-s) y(s), y(s) \right\rangle ds + \sum_{j=1}^{p} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt} K_j(t-s) y(s), y(s) \right\rangle ds.\end{aligned}$$

Therefore, taking into account the definition of C, we obtain

$$\begin{aligned}\frac{d}{dt}V(t,y) + &\left\langle C \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix} \right\rangle - \int_{t-\tau}^{t} \left\langle \frac{d}{dt} K(t-s) y(s), y(s) \right\rangle ds \\ &- \sum_{j=1}^{p} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt} K_j(t-s) y(s), y(s) \right\rangle ds \equiv 0. \qquad (11)\end{aligned}$$

According to the conditions of the Theorem, the matrix C is nonnegative definite and satisfies (8). Hence,

$$\left\langle C \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix} \right\rangle \geq c(\|y(t)\|^2 + \|y(t-\tau)\|^2).$$

Therefore, by analogy with [10], we have the inequality

$$\left\langle C \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau) \\ y(t-\tau_1) \\ \vdots \\ y(t-\tau_p) \end{pmatrix} \right\rangle \geq \frac{c}{(1+\|D\|^2)\|H\|} \langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle.$$

Now, taking into account the conditions on the matrices $K(s)$ and $K_j(s)$ from (11), we obtain the estimate

$$\begin{aligned}\frac{d}{dt}V(t,y) + &\frac{c}{(1+\|D\|^2)\|H\|} \langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &+ k \int_{t-\tau}^{t} \langle K(t-s) y(s), y(s)\rangle\, ds + \sum_{j=1}^{p} k_j \int_{t-\tau_j}^{t} \langle K_j(t-s) y(s), y(s)\rangle\, ds \leq 0.\end{aligned}$$

Then, by (10), we have

$$\frac{d}{dt}V(t,y) + \frac{\gamma}{\|H\|}V(t,y) \le 0, \qquad t \ge \tau.$$

Therefore,

$$V(t,y) \le \exp\left(-\frac{\gamma(t-\tau)}{\|H\|}\right) V(\tau,\varphi), \qquad t \ge \tau.$$

According to the definition of (6), this inequality coincides with (9).

The proof of Theorem 2 is complete. □

Corollary 1. *Let the conditions of Theorem 2 be satisfied. Then for a solution $y(t)$ to (5) the estimate holds*

$$\|y(t) + Dy(t-\tau)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}, \qquad t > \tau, \tag{12}$$

where $\gamma > 0$ is defined by (10).

The proof is immediate from (9) and the definition of (6).

3. ESTIMATES OF SOLUTIONS TO EQUATIONS OF NEUTRAL TYPE

The estimates, established in Section 2, can be used by analogy with [10] for proving an asymptotic stability of the zero solution to the system of equations of neutral type (1) and obtaining some estimates of the solutions. For this, we need the following discrete analogs of the Lyapunov and Krein theorems on the properties of solutions to the discrete Lyapunov equation:

$$\widetilde{H} - D^*\widetilde{H}D = I \tag{13}$$

(for instance, see [11, 12]).

Theorem 3. *Let the spectrum of the matrix D be in the unit circle. Then there exists a unique solution $\widetilde{H} = \widetilde{H}^* > 0$ to* (13) *and*

$$\|D^l\| \le (1 - 1/\|\widetilde{H}\|)^{l/2}\sqrt{\mu(\widetilde{H})}, \qquad l = 1, 2, \dots, \tag{14}$$

where $\mu(\widetilde{H})$ is the condition number of $\widetilde{H}$.

Using Corollary 1, we give some estimates for the norm of a solution to the initial value problem (5).

Theorem 4. *Let the conditions of Theorem 2 be satisfied and let $\widetilde{H} = \widetilde{H}^* > 0$ be a solution to the equation* (13). *Then for a solution $y(t)$ to* (5) *the following take place:*

(a) *if $(1 - 1/\|\widetilde{H}\|)^{1/2} < q^{-1}$ with $q = \exp(\gamma\tau/(2\|H\|))$ then*

$$\|y(t)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\mu(\widetilde{H})}\left(\frac{\sqrt{\|H^{-1}\|\,V(\tau,\varphi)}}{1 - (1 - 1/\|\widetilde{H}\|)^{1/2}q}\right) + (1 - 1/\|\widetilde{H}\|)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi, \qquad t > \tau, \tag{15}$$

where $\Phi = \max\limits_{s\in[0,\tau]}\|\varphi(s)\|$;

(b) *if $(1 - 1/\|\widetilde{H}\|)^{1/2} = q^{-1}$ then*

$$\|y(t)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\frac{t}{\tau}\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|\,V(\tau,\varphi)} + (1 - 1/\|\widetilde{H}\|)^{(t-\tau)/2\tau}\sqrt{\mu(\widetilde{H})}\Phi, \qquad t > \tau; \tag{16}$$

(c) *if* $q^{-1} < (1 - 1/\|\widetilde{H}\|)^{1/2}$ *then*

$$\|y(t)\| \le \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)} \sqrt{\mu(\widetilde{H})} \left(\frac{\sqrt{\|H^{-1}\|\, V(\tau,\varphi)}}{1-(1-1/\|\widetilde{H}\|)^{-1/2}q^{-1}} + \Phi\right), \qquad t > \tau. \tag{17}$$

Proof. Let $Y(t) = y(t) + Dy(t-\tau)$. Then, obviously, for $t \in [l\tau, (l+1)\tau)$, the vector-function $y(t)$ can be written as

$$y(t) = Y(t) - DY(t-\tau) + \dots + (-1)^{l-1}D^{l-1}Y(t-(l-1)\tau) + (-1)^l D^l \varphi(t - l\tau).$$

Hence, we have the estimate

$$\|y(t)\| \le \|Y(t)\| + \|D\| \cdot \|Y(t-\tau)\| + \dots + \|D^{l-1}\| \cdot \|Y(t-(l-1)\tau)\| + \|D^l\|\Phi.$$

Therefore, using the inequalities (12) and (14) we obtain

$$\begin{aligned}\|y(t)\| \le{}& \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} \\ &+ \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{1/2}\sqrt{\mu(\widetilde{H})}\exp\left(-\frac{\gamma(t-2\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} \\ &+ \dots + \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{(l-1)/2}\sqrt{\mu(\widetilde{H})}\exp\left(-\frac{\gamma(t-l\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} \\ &+ \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{l/2}\sqrt{\mu(\widetilde{H})}\Phi,\end{aligned}$$

and then we have

$$\begin{aligned}\|y(t)\| \le{}& \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(\sum_{j=0}^{l-1}\left(\left(1-\frac{1}{\|\widetilde{H}\|}\right)^{1/2}q\right)^j\right)\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|\,V(\tau,\varphi)} \\ &+ \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\,\Phi, \qquad t\in[l\tau,(l+1)\tau).\end{aligned} \tag{18}$$

Consider the first case: $(1 - 1/\|\widetilde{H}\|)^{1/2} < q^{-1}$. Represent t as $t = l\tau + s$, where $s \in [0, \tau)$. Then, from the estimate (18) we infer

$$\begin{aligned}\|y(t)\| \le{}& \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(\sum_{j=0}^{\infty}\left(\left(1-\frac{1}{\|\widetilde{H}\|}\right)^{1/2}q\right)^j\right)\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|\,V(\tau,\varphi)} \\ &+ \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi \\ ={}& \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|\,V(\tau,\varphi)}\left(\frac{1}{1-(1-1/\|\widetilde{H}\|)^{1/2}q}\right) \\ &+ \left(1-\frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi,\end{aligned}$$

i.e., (15) is proved.

Consider the second case: $(1 - 1/\|\widetilde{H}\|)^{1/2} = q^{-1}$. Let $t = l\tau + s$, where $s \in [0, \tau)$. Then, obviously, from (18) we obtain

$$\|y(t)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|V(\tau,\varphi)}\,l + \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi.$$

The inequality (16) is immediate from this estimate.

Consider now the third case: $q^{-1} < (1 - 1/\|\widetilde{H}\|)^{1/2}$. Representing $t = l\tau + s$ with $s \in [0, \tau)$, from the estimate (18) we have

$$\begin{aligned}\|y(t)\| &\le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(\left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{1/2} q\right)^{l-1}\left(\sum_{j=0}^{\infty}\left(\left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{1/2} q\right)^{-j}\right)\\ &\qquad\times\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|V(\tau,\varphi)} + \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi\\ &= \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(\left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{1/2} q\right)^{l}\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|V(\tau,\varphi)}\\ &\qquad\times\left(\frac{1}{(1 - 1/\|\widetilde{H}\|)^{1/2}q - 1}\right) + \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi.\end{aligned}$$

Hence, taking into account $0 \le s < \tau$, we obtain

$$\begin{aligned}\|y(t)\| &\le \exp\left(\frac{\gamma(\tau-s)}{2\|H\|}\right)\left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{l/2}\sqrt{\mu(\widetilde{H})}\sqrt{\|H^{-1}\|V(\tau,\varphi)}\\ &\qquad\times\left(\frac{1}{(1 - 1/\|\widetilde{H}\|)^{1/2}q - 1}\right) + \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\Phi\\ &\le \left(1 - \frac{1}{\|\widetilde{H}\|}\right)^{(t-\tau)/(2\tau)}\sqrt{\mu(\widetilde{H})}\left(\frac{\sqrt{\|H^{-1}\|V(\tau,\varphi)}}{1 - (1 - 1/\|\widetilde{H}\|)^{-1/2}q^{-1}} + \Phi\right),\end{aligned}$$

i.e., the inequality (17) is proved.

The proof of Theorem 4 is complete. □

Corollary 2. *Let the conditions of Theorem* 4 *be satisfied. Then the zero solution to the system of equations* (1) *is asymptotically stable.*

The proof is immediate from (15)–(17) and the definition (6).

Remark. As it follows from (15)–(17) solutions to (1) decrease with an exponential rate as $t \to \infty$.

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